SCATTERING OF THE CAPSULE LANDING POINTS

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1. FORMULATION OF THE PROBLEM

At implementation of descent of capsules from orbit as a result of influence of the different perturbed factors to motion the scattering of points of a landing - deviation of actual points of a landing from computational takes place. On the value of scattering is influenced with such factors as the errors arising at separation of capsules, the errors caused by perturbations on the atmospheric leg of the trajectory.

We shall consider the motion of capsule on an extra-atmospheric leg of the trajectory and then at a drop in dense atmospheric slices.

Knowing errors of definition of coordinates and projections of velocities of capsule at its separation does not call difficulties the solution of a problem of definition of the scattering caused by the data by the perturbed factor.

At the motion of a capsule from the moment of separation up to altitudes about 100 km its the angular motion - motion about the center of mass is determined in basic by initial values of angles and angular velocity, by influence of the gravitational and aerodynamic moments. Owing to what it is necessary to expect, that the angles of attack (angle of attack - angle between a direction of the capsule longitudinal axis and the mass center velocity vector) can have any value.

At motion of the capsule on an extra-atmospheric leg of trajectory, considering a small time interval, it is possible to neglect the external moments, owing to what the motion of the capsule is determined by the laws of motion of a rigid body in case of the Euler.

We shall consider a case when capsule is axisymmetric. Then the rotary motion of the capsule on an extra-atmospheric leg of trajectory represents regular precession at which the longitudinal axis passing through a center of mass, describes a circular cone concerning constant in the space directions of the kinetic moment vector $\vec{K}_0$ (see Fig. 1, where α_k is the angle between a direction of the capsule longitudinal axis and the kinetic moment vector, α_V is the angle between a direction of the kinetic moment vector and the mass center velocity vector, α_s is the angle between a direction of the capsule longitudinal axis and the mass center velocity vector - spatial angle of attack, ψ is the angle of extra-atmospheric precession).

If the initial kinetic moment is directed so, that the motion appears flat, then at motion outside of atmosphere the capsule rotates with constant velocity around of a transversal axis.

After transiting an extra-atmospheric leg of the trajectory the capsule starts to be descended in atmosphere and in process of increase of density all in the greater and greater degree starts to experience a stabilizing operation of atmosphere (rigidity of a system is incremented on some orders). The indicated effect of variability of parameters in a system is a major factor determining fading of oscillations. The influence of variability of parameters in a system is the same important singularity of a considered problem, as well as presence of large angles of attack.

At motion of unguided capsules in atmosphere it is possible to select three basic groups of the perturbed factors calling deviation of points of a landing from computational: first - dispersion of thermodynamic parameters of atmosphere and wind, second - deviation of a ballistic coefficient, third - presence of a ascensional aerodynamic force and its space orientation.

Now problem of definition of computational scattering of points of a landing at an operation of two first groups of the perturbed factors is explored full enough. On the indicated perturbations there are relevant standards, there are effective techniques of calculation of their influence. In basic complexity of fulfilment of a problem of definition of scattering of points of a landing is determined by presence of a ascensional aerodynamic force, the orientation by which one in space, as a rule, has random character. The influence of the ascensional aerodynamic force on scattering of points of a landing is determined to character of motion of the capsule about the

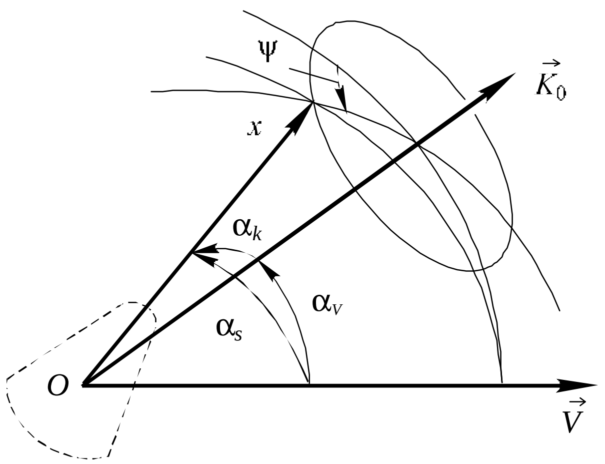

Fig. 1. Parameters defining out atmosphere motion of capsule.

mass center.

In case at atmospheric entry the motion of the capsule about mass center appears planar and rotary, at transition in an oscillating motion the cases of capsule hovering in the unstable equilibrium vicinity on an angle of attack are possible, that can result in considerable deviation of points of a landing from computational [1]. Because of small asymmetry of the capsule implementation of an autorotation of the capsule - long rotation without transition in oscillations is possible also.

In case of spatial motion about the mass center the operation of a ascensional aerodynamic force on scattering on the average during a complete revolution of the capsule longitudinal axis about the mass center velocity vector will be equal to null, if the spatial angle of attack remains to stationary values or changes insignificantly, and capsule precesses about velocity vector from a stationary and large on value by a velocity. The graphics image of the relevant spiral precessional movement of the capsule at descent in atmosphere is adduced on Fig. 2 - nominal

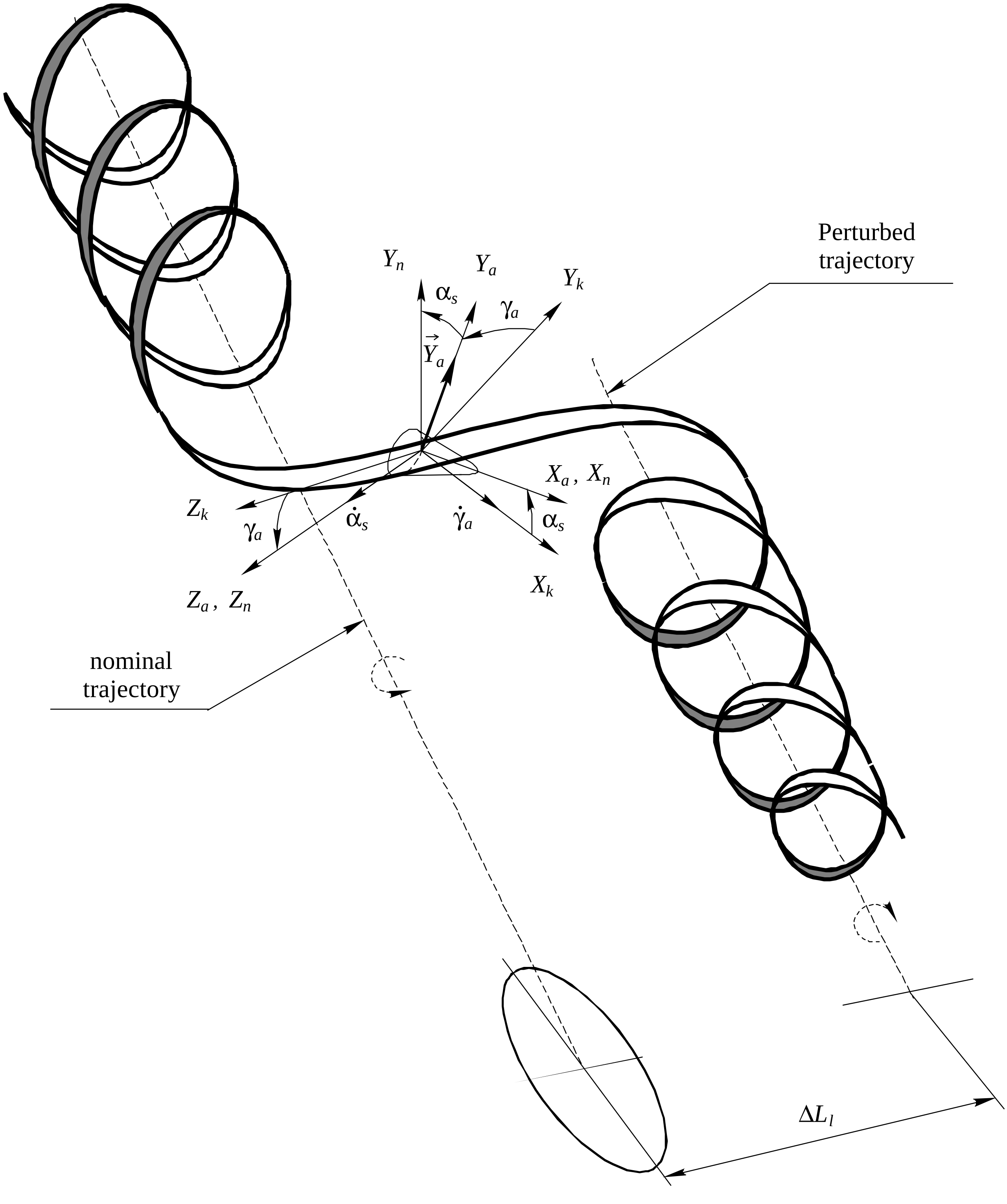

Fig. 2. Precessional motion capsule on the atmospheric leg of the trajectory.

trajectory

Situations when conditions called above are not implemented are actually possible. In outcome there is a lateral scattering - deviation from a nominal trajectory perpendicularly planes of a trajectory caused by an operation of ascensional aerodynamic force, which one for small time intervals approximately can be estimated as [2]:

$$\Delta L_l \approx r\frac{C_{Ya}^{\alpha} qS}{mV}\int_{t_0}^{t}\alpha_s e^{i\int\dot{\gamma}_a dt}dt,$$

where r is the distance from a point, in which one has worked a perturbation, up to a surface of the Earth; q is the velocity head; m is the capsule mass; V -is the mass center velocity; S is the characteristic area of the capsule; C_{Ya}^{α} is the derivative of the coefficient of ascensional aerodynamic force on an angle of attack; α_s is the spatial angle of attack; γ_a is the angle of precession - angle between a plane of flight and plane of a spatial angle of attack; $\dot{\gamma}_a$ is the angular velocity precession.

The articles [2, 3, 4, 5, 6, 7] are dedicated to the analysis of such situations. It is possible to select a series of cases, at which lateral scattering is implemented: transition of longitudinal angular rate - angular rate of rolls through a zero [4, 5, 6]; origin of the small pulse moment connected to the body of the capsule [2, 6]; origin of the small pulse moment connected to a ram airflow [2, 6]; sharp change of the balancing angle of attack/6/.

The most considerable deviation of a point of a landing from the purpose arises at transition of angular rate of a roll ω_x through a zero. At motion of the capsule in atmosphere balancing component of an angle of attack α_0 caused by aerodynamic, mass and inertial asymmetry, in trajectory coordinate system is determined by the following formula

$$\delta_0 = \alpha_0 \exp(i\int_{t_0}^{t}\omega_x dt).$$

At the moment of transition of angular rate through a zero the vector of ascensional aerodynamic force caused by a balancing angle of attack α_0, will be stationary concerning inertial space, that will lead to a deviation of capsule from a nominal trajectory and deviation of a point of a landing from computational (see Fig. 2 - perturbed trajectory).

At descent in atmosphere on the capsule the different perturbed factors on a roll act, from which one basic are:

- the moments stipulated by initial asymmetry of the capsule, caused technological error of manufacturing;
- the moments caused by asymmetry, arising during asymmetrical scorching of heat-shielding cover;
- the moments stipulated by effects of viscous interaction of a ram airflow with a relief of a lateral area;
- the damping moments.

The indicated perturbed moments can result to damping angular rate of a roll - to transition of its value through a zero, and, taking into account their random character, and also that the value of angular rate of a roll on atmospheric entry is a small, of such transitions can be a several. Thus the value of deflection of a point of a landing from the purpose can make some kilometers.

It is necessary to mark, that angular behaviour of capsule about a ram airflow and character of change of its angular rate influence not only on accuracy of definition of points of a landing but also on arrangement and weight of heat-shielding cover, on values of the dynamic loads which are operational on capsule.

It is necessary to mark also as the design uncontrolled capsule guesses absence of any control objects which are capable to control by angular of motion, the maintenance of a computational conditions of motion such capsule implements only on a design stage by selection of design-ballistic parameters.

With the purposes of fulfilment of the functional requirements installed to descent capsules, on their some inertial and geometrical characteristics a series of limitations is fix. As a rule, at designing and manufacturing of such capsules aim to ensure a dynamic symmetry and also axisymmetrical shape of an external surface. The last circumstance results that a basic component of the external aerodynamic moment which is operational on the capsule, is the so-called restoring moment, the plane of an operation which one coincides with a plane of a spatial angle of attack, and the value depends on value of this angle.

The situation, analogous from the point of view of a mechanics, takes place in a classic case of the Lagrange of motion of a heavy rigid body, when the alone external moment created in gravity, acts in a vertical plane and on value is proportional to a sine of a angle of notation. However there is also relevant difference consisting that the character of dependence of the restoring aerodynamic moment from an angle of attack, defined geometrical configuration of the capsule and flight phases, can be enough composite and differ from sinusoidal. Besides the shape of this dependence usually changes along flight trajectory pursuant to parameters of a ram airflow.

The sinusoidal dependence of the restoring aerodynamic moment to an angle of attack is characteristic for capsules representing an orb or a slender cone. Now are designed and the descent capsules having segmentally-conical, blunted conical and other shapes of an external surface (descent modules «Soyuz», «Mars» many perspective small-sized cargo capsules), with enough composite aerodynamic characteristics are operated also, for satisfactory approximation which one by trigonometric series in last are necessary retain not less than two harmonicses.

The marked singularities of a perturbed motion of the capsule at atmospheric entry allow to conclude, that the descent capsule represents essentially non-linear mechanical system with variable parameters.

The motion of the capsule in atmosphere as rigid body is described by a system of non-linear differential equations 12-th order with floating coefficients, the general solution by which one is not obviously possible for receiving. At a numerical integration of equations of motion, at first, there are latent reasons conditioning this or that character of motion, secondly, for installation of regularities of motion the considerable number of calculations is required, and it even with usage of modern fast-response computers results in large costs of time because of presence in right members of equations of fast oscillating functions. Searching the approximate analytical solutions both development of mathematical models and research techniques therefore is rather actual permitting it is essential to speed up process of calculation and to establish regularities, appropriute motion of a body.

We shall consider two problems: about investigation of the precessional motion of the capsule and about investigation of transient modes of capsule angular motion on the upper leg of the reentry trajectory

2. INVESTIGATION OF THE PRECESSIONAL MOTION

Let us consider the motion about the mass center of an axisymmetric capsule on the initial atmospheric leg of the trajectory.

After atmospheric entry the statically steady capsule starts to experience an operation of the restoring aerodynamic moment, which aims to combine a longitudinal axis with the mass center velocity vector. However pitching motion is counteracted by gyroscopic forces calling an ordered precession of the kinetic moment vector concerning the mass center velocity vector. The kinetic moment vector deviates in that party where the vector of a restoring aerodynamic moment is directed.

Figure 3 shows the various cases of rotational motions of the axisymmetric capsule on the initial atmospheric leg of the trajectory, the projections of trajectories of nose point of the capsule onto the plane which perpendicular to the mass center velocity vector. In a problem about descent of a spacecraft in atmosphere the following nomenclature is adopted. The precession of the spacecraft longitudinal axis about the mass center velocity vector, on period equal to phase of the complete revolution, on a direction to the given vector (for a case $\omega_x > 0$), it is accepted to name as inverse precession (see Figs. 3, *a* and *b*), and conterminous to a direction of mass center velocity vector $\vec{V}$ - direct precession (see Figs. 3, *c* and *d*) [8].

When studying the uncontrolled spatial motion of a capsule in the atmosphere, problems associated with the behavior of the amplitude values of the angle of attack are usually considered; this angle can often be determined by asymptotic methods independently of the other angles, i.e., the angle of precession and the angle of proper rotation. However, in the problems of scattering of the capsule landing points and problems of scorching of the heat shield, there arises a question of investigation of the precessional motion. We will present a very simple grapho-analytical scheme for analysis of the precessional motion by the amplitude characteristics of the angle of attack.

The solution of a linearized equation of motion; this equation, written for the complex angle of attack in the trajectory frame of reference [2], in the case of nonresonant motion, has the following form (for uniqueness, we assume $\omega_x > 0$):

$$\delta = \alpha_k(t)\exp(i\psi_1) + \alpha_v(t)\exp(i\psi_2) + \alpha_0(t)\exp(i\psi_0) \qquad (1)$$

where

$$\psi_1 = \int_{t_0}^{t} (\omega + 0.5\bar{I}_x\omega_x)dt,$$

$$\psi_2 = \int_{t_0}^{t} (-\omega + 0.5\bar{I}_x\omega_x)dt, \quad \psi_0 = \int_{t_0}^{t} \omega_x dt, \qquad (2)$$

$$\alpha_k(t) = \alpha_k(t_0)\exp[-\int_{t_0}^{t} 0.5 d_1 dt]\exp[\int_{t_0}^{t} d_3 dt]\sqrt{\omega(t_0)/\omega(t)},$$

$$\alpha_v(t) = \alpha_v(t_0)\exp[-\int_{t_0}^{t} 0.5 d_1 dt]\exp[-\int_{t_0}^{t} d_3 dt]\sqrt{\omega(t_0)/\omega(t)},$$

$$d_1 = (C_n^{\alpha} - C_{\tau})qS/(mV) - m^{\omega}qSl^2/(IV),$$

$$d_3 = [(0.5\bar{I}_x(C_n^{a} - C_t + m^{w}ml^2/I) - m_m ml^2/I)\cdot$$
$$\cdot w_x qS/(mV) - 0.5\bar{I}_x\dot{w}_x]/(2w)$$

,

$\alpha_k(t)$, $\alpha_v(t)$ is the amplitude characteristics of the angle of attack which, on the extra-atmospheric leg of the flight, characterize the position of the capsule longitudinal axis with respect to the kinetic moment vector $\vec{K}$ and position of the kinetic moment vector $\vec{K}$ with respect to the mass center velocity vector $\vec{V}$, respectively; and, on the atmospheric leg of the flight, they are slowly varying functions (their variation rate is of the order of ε); $\alpha_0(t)$ is the balancing angle of attack, caused by a small inertial-aerodynamic asymmetry of the capsule;

$\omega = [C_n^{\alpha}(\bar{x}_p - \bar{x}_g)qSl/I + (\bar{I}_x\omega_x)^2/4]^{1/2}$ is the frequency of proper oscillations of the capsule; $\bar{I}_x = I_x/I$; I_x and I are the longitudinal and transversal moments of inertia, respectively; l is the characteristic size of the capsule; $(\bar{x}_p - \bar{x}_g)$ is the statical stability factor; C_n^{α} is the derivative of the coefficient of aerodynamic normal force; C_{τ} is the coefficient of aerodynamic tangential force; m^{ω} is the coefficient of damping moment about the transverse axis; m_m is the coefficient of the moment of Magnus.

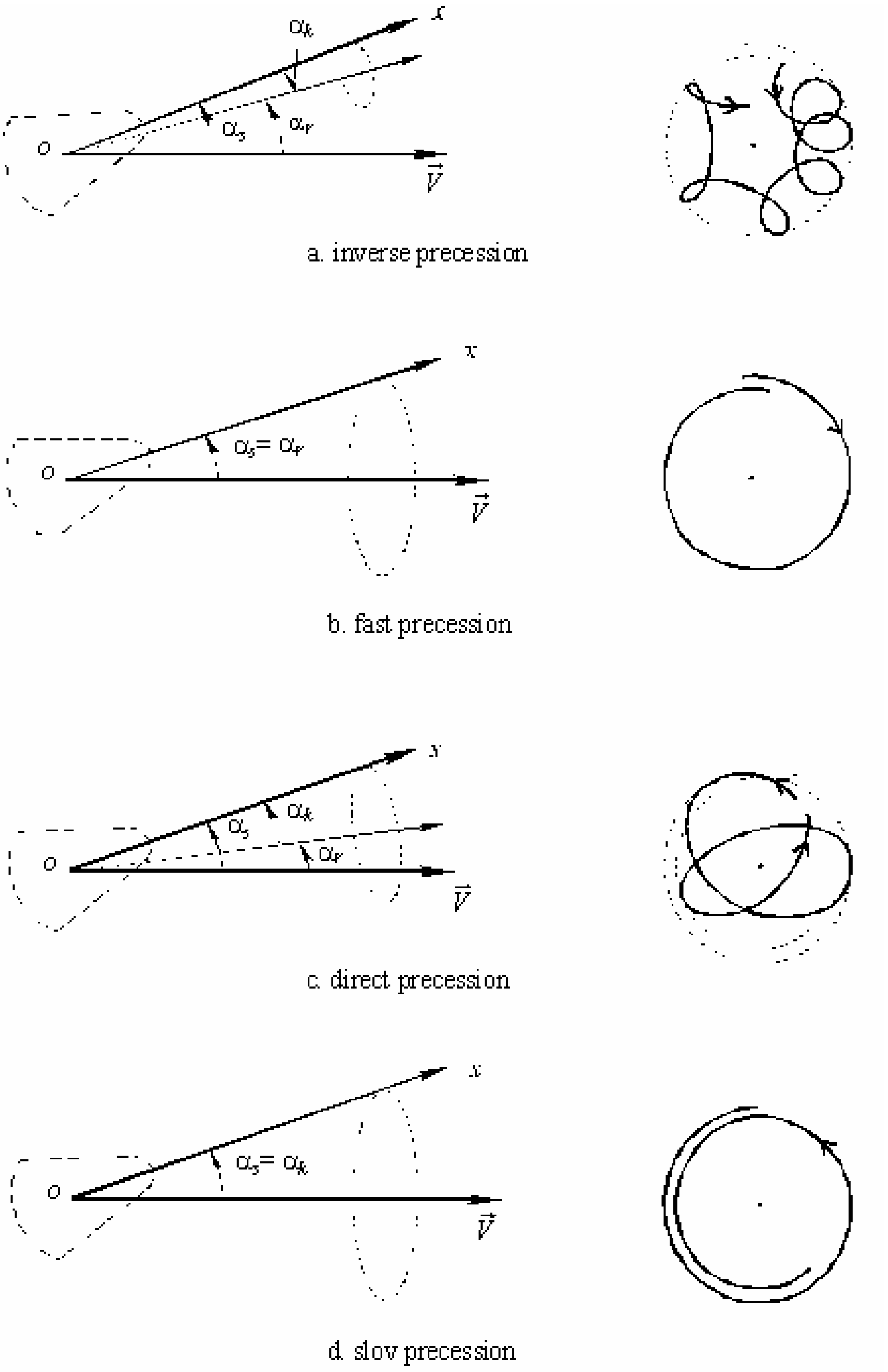

Fig. 3. Possible precessional motions of the axissymmetric capsule on the initial atmospheric leg of the trajectory.

The formula (1) displays, that motion of a nose point of the capsule, concerning a center of mass describes some curve being a superposition of three rotary motions. First from them happens on a round of radius α_k and has instant angular velocity $\dot{\psi}_1 = \omega + 0.5\bar{I}_x\omega_x$. It happens around of a point which in turn goes on a round of radius α_v with instant angular velocity $\dot{\psi}_2 = -\omega + 0.5\bar{I}_x\omega_x$.

And this motion happens around of a point which goes on a round of radius α_0 with instant angular velocity $\dot{\psi}_3 = \omega_x$. Such introducing of an angle of attack as the sum of three vectors is figured on Fig. 4.

Let us express the complex angle of attack (1) in terms of projections onto the axes OY_k and OZ_k of the trajectory frame of reference,

$$\delta_{yk} = \alpha_k \cos\psi_1 + \alpha_v \cos\psi_2 + \alpha_0 \cos\psi_0 \,,$$

$$\delta_{zk} = \alpha_k \sin\psi_1 + \alpha_v \sin\psi_2 + \alpha_0 \sin\psi_0 \,.$$

Then the expressions for modulus of the angle of attack and velocity bank angle (angle of precession), which, in this case, is the phase angle, can be rewritten as follows:

$$d^2 = a_s^2 = a_k^2 + a_v^2 + a_0^2 + 2a_k a_v \, cos(y_2 - y_1) +$$

$$+ 2\alpha_k \alpha_0 \cos(\psi_1 - \psi_0) + 2\alpha_v \alpha_0 \cos(\psi_2 - \psi_0)\,, \quad (3)$$

$$\gamma_a = \arctan[\delta_{zk} / \delta_{yk}]\,. \quad (4)$$

We differentiate (17) to obtain the angular velocity of precession, accurate up to the values of the order of ε,

$$\dot{g}_a^2 = [\dot{y}_1 a_k^2 + \dot{y}_2 a_v^2 + \dot{y}_0 a_0^2 + (\dot{y}_1 + \dot{y}_2)\cdot$$
$$\cdot a_k a_v \, cos(y_2 - y_1) + (\dot{y}_1 + \dot{y}_0) a_k a_0 \cdot$$
$$\cdot cos(y_1 - y_0) + (\dot{y}_2 + \dot{y}_0) a_v a_0 \cdot$$
$$\cdot cos(y_2 - y_0)] / a_s^2$$

Extremal values of the angular velocity of precession $\dot{\gamma}_a$ correspond to the extremal values of the modulus of the spatial angle of attack and, in view of (2) and (3), they are

$$\dot{g}_a(a_{max}) = [(a_k + a_v)\bar{I}_x w_x / 2 + (a_k - a_v) w + +a_0 w_x] / (a_k + a_v + a_0) \;;$$

$$\dot{\gamma}_a(\alpha_{min}) = \begin{cases} [(\alpha_v - \alpha_k)\bar{I}_x\omega_x/2 - (\alpha_k + \alpha_v)\omega + \alpha_0\omega_x]/(\alpha_v - \alpha_k + \alpha_0) \\ \text{at } \alpha_k > \alpha_v, \ \alpha_k > \alpha_0, \\ [(\alpha_k - \alpha_v)\bar{I}_x\omega_x/2 + (\alpha_k + \alpha_v)\omega + \alpha_0\omega_x]/(\alpha_k - \alpha_v + \alpha_0) \\ \text{at } \alpha_v > \alpha_k, \ \alpha_v > \alpha_0, \\ [(\alpha_k + \alpha_v)\bar{I}_x\omega_x/2 + (\alpha_k - \alpha_v)\omega - \alpha_0\omega_x]/(\alpha_k + \alpha_v - \alpha_0) \\ \text{at } \alpha_0 > \alpha_k, \ \alpha_0 > \alpha_v, \end{cases} \quad (5)$$

For the axisymmetric capsule, the extremal values of the angular velocity of precession $\dot{\gamma}_a$ are defined by the expressions

$$\dot{\gamma}_a(\alpha_{max,min}) = \bar{I}_x\omega_x / 2 + [(\alpha_k \mp \alpha_v)/(\alpha_k \pm \alpha_v)]\omega \quad (6)$$

for atmospheric leg of trajectory and

$$\dot{\gamma}_a(\alpha_{max,\,min}) = \alpha_k(\bar{I}_x\omega_x / 2)/(\alpha_k \pm \alpha_v)\,, \quad (7)$$

for the extra-atmospheric leg.

Note that from the equation for the angular velocity of proper rotation $\dot{\varphi}_s = \omega_x - \dot{\gamma}_a$ one can determine the extremal values of the angular velocity

$$\dot{\varphi}_s(\alpha_{max,\,min}) = \omega_x - \dot{\gamma}_a(\alpha_{max,\,min})\,. \quad (8)$$

Using the expressions for the extremal values of $\dot{\gamma}_a$ and $\dot{\varphi}_s$ (7) - (8), one can plot the regions of possible values of the angular velocity of precession and angular velocity of proper rotation of the capsule against the ratio of amplitude characteristics of the angle of attack.

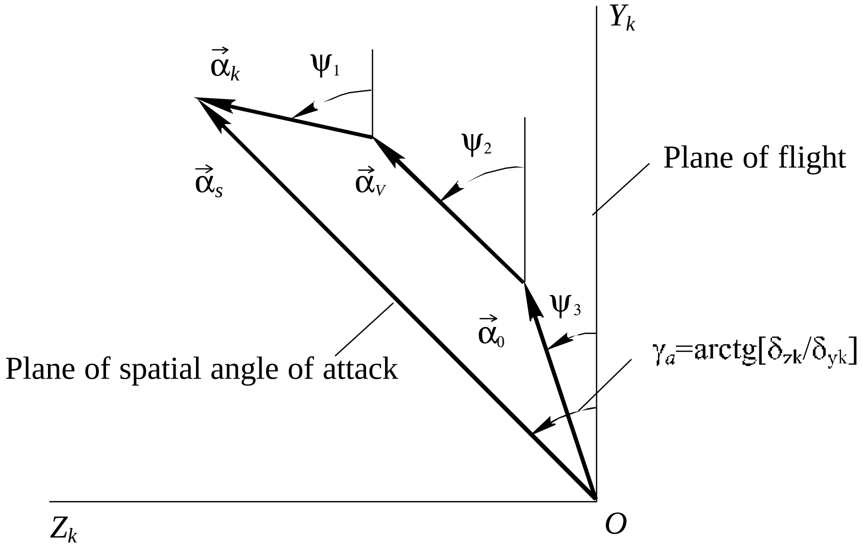

Fig. 4. Complex angle of attack $\delta = \alpha_s \exp(i\gamma_a)$ as a sum of three vectors.

Figure 5 shows, as an example, the regions of probable precessional motions of the axisymmetric capsule on the atmospheric leg of trajectory in relation to the ratio of amplitude characteristics of the angle of attack α_k / α_v and α_v / α_k. Under the condition $\alpha_v > \alpha_k$, the angular velocity of precession oscillates about the value $\bar{I}_x \omega_x / 2 - \omega$. This kind of precessional motion,

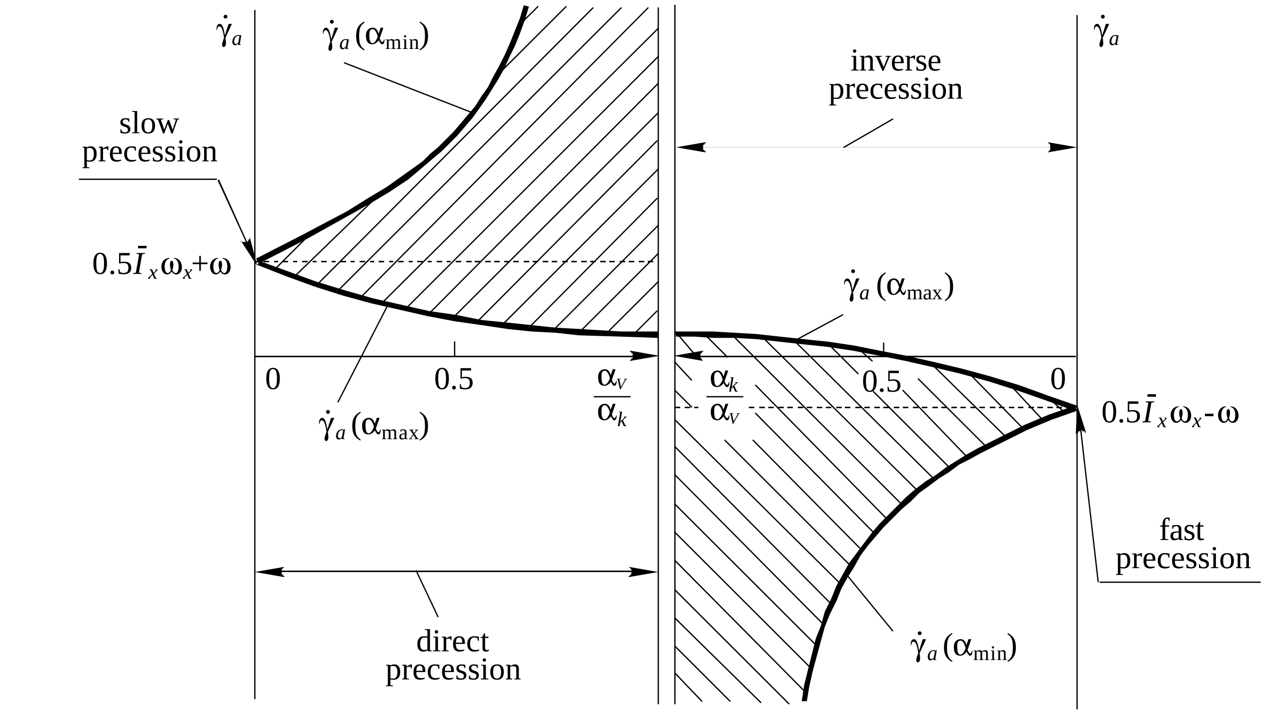

Fig. 5. Region of possible precessional motions of the axisymmetric capsule on the atmospheric leg of the trajectory.

according to [8], is called inverse precession. Under the condition $\alpha_v < \alpha_k$, the angular velocity of precession oscillates about the value $\bar{I}_x \omega_x / 2 + \omega$. This kind of precessional motion is called direct precession [8]. Limiting cases of the rotational motion, where the capsule executes the symmetric conic motion $\alpha_k = 0$, $\alpha_v \neq 0$ or $\alpha_v = 0$, $\alpha_k \neq 0$, are referred to as fast and slow precessions, respectively, [1].

Figure 6 displays the regions of possible precessional motions of the capsule with a small asymmetry, on the atmospheric leg of trajectory, in relation to the ratio of amplitude characteristics of the angle of attack α_v / α_0 and α_0 / α_v, for $\alpha_k = 0$. At $\alpha_0 > \alpha_v$, the angular velocity of precession oscillates about the value of the longitudinal angular velocity ω_x (the capsule executes oscillations in the angle of proper rotation). The limiting case of the rotational motion where the angular velocity of precession is equal to the longitudinal angular velocity of the capsule (the angular velocity of proper rotation is zero) is referred to as "lunar" motion (the capsule always faces the main stream by the same side) [7].

In the resonant case, investigation of the precessional motion can be pursued with the use of the relationship (6); here, however, the amplitude characteristic α_k defined by the slow precession should be substituted for by the geometrical sum of this characteristic and the induced balancing component of the angle of attack, their rotation frequencies being close to each other [1].

In the case where the three amplitude characteristics of the angle of attack are nonzero ($\alpha_k \neq 0$, $\alpha_v \neq 0$, $\alpha_0 \neq 0$), the graphical representation of the region of possible values of the angular velocity of precession loses the simplicity and clearness. In this case, the analysis of the precessional motion can be accomplished with the use of the relationships (5).

The proposed scheme for investigation of the precessional motion allows us, provided the solution to the equations for the angle of attack is known, to find the range of variation of the angular velocity of precession, to determine the type of the precessional motion, and to study the nature of its variation as a function of the ratio of amplitude characteristics of the angle of attack.

3. TRANSIENT MODES OF CAPSULE ANGULAR MOTION ON THE UPPER SECTION OF THE RE-ENTRY TRAJECTORY

Let us consider the motion of an axisymmetric capsule about its center of mass on the upper section of the atmospheric reentry trajectory. In this case, the variation of the velocity of the center of mass, the trajectory inclination angle, and the aerodynamic damping may be neglected. We investigate cases in which the character of motion changes during the reentry process: the rotational motion transfers into an oscillatory one, and the oscillatory motion transfers "by jumping" into the oscillatory motion with other amplitude characteristics.

For axisymmetric capsules, the coefficients of aerodynamic forces and moments can be represented in the form trigonometric series. If the capsule has the shape of an orb or slender cone, its the moment characteristic (the relation of a restoring aerodynamic moment to the transversal moment of inertia) is described by sinusoidal angle of attack dependence (Fig. 7):

$$M_\alpha(\alpha) = m_\alpha qSl / I = a\sin\alpha,$$

where m_α is the coefficient of restoring moment, $\alpha \equiv \alpha_s$ is the spatial angle of attack.

For capsules segmentally-conical, blunted conical shapes the appearance of three balance positions on an angle of attack is possible. Apparently, those in this case for satisfactory approximation of dependence of the moment characteristic from of an angle of attack retain not less than two harmonicses of a trigonometric series in expansion (Fig. 8.):

$$M_\alpha(\alpha) = a\sin\alpha + b\sin 2\alpha. \qquad (9)$$

The motion of a capsule with biharmonic moment characteristics (9) about the center of mass under the aforementioned assumptions is described by the system with slowly varying parameters of type [1, 10]

$$\ddot{\alpha} + F(\alpha) = 0, \qquad (10)$$

$$F(a) = (G - R\cos a)(R - G\cos a)/\sin^3 a + \\ + a\sin a + b\sin 2a,$$

$$a = a(z), \quad b = b(z),$$

where $R = \text{const}$, $G = \text{const}$ are projections of a kinetic moment vector on the longitudinal axis of a capsule and onto the direction of the mass center velocity, normalized with respect to the transversal moment of inertia; $a(z)$, $b(z)$ are moment characteristic coefficients; z is a slowly varying parameter.

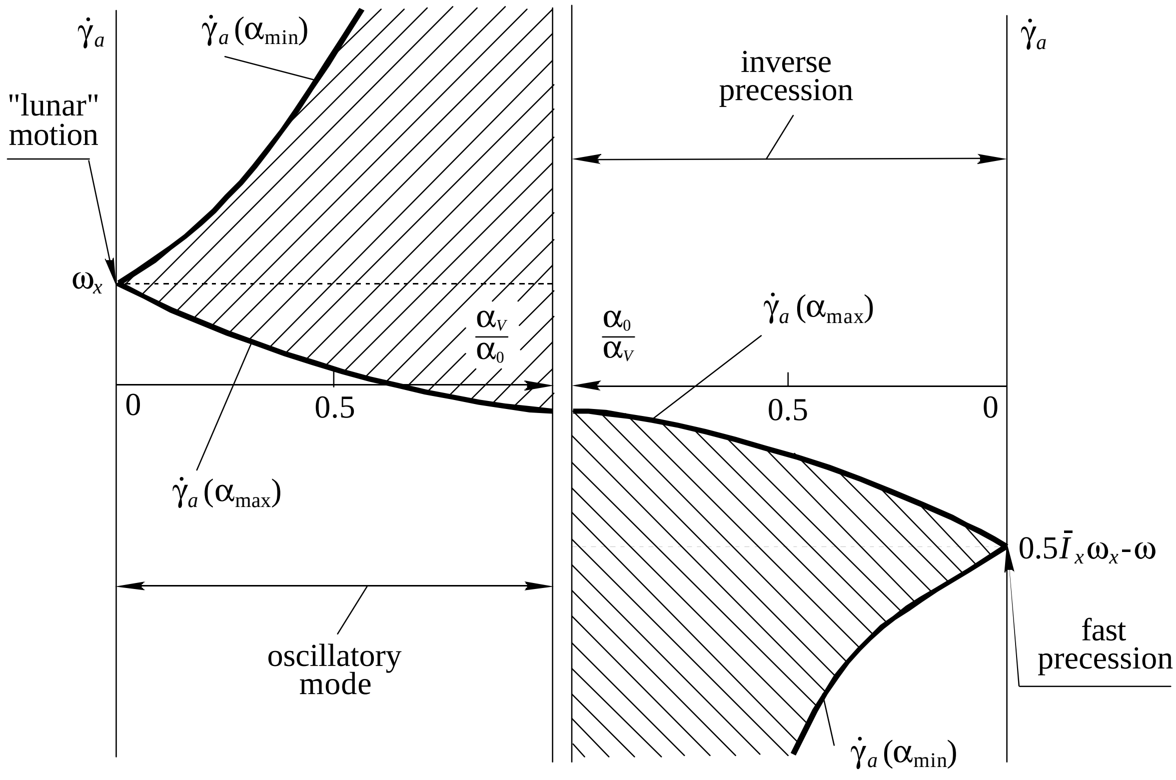

Fig. 6. Region of possible precessional motions of the capsule with a small asymmetry on the atmospheric leg of the trajectory (α_k=0).

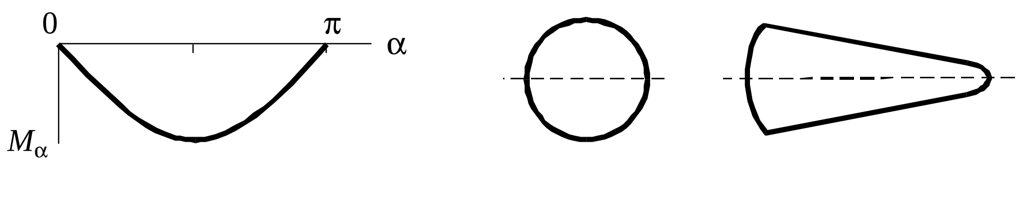

Fig.7. The sinusoidal moment characteristic for the capsule having the shape of an orb or slender cone.

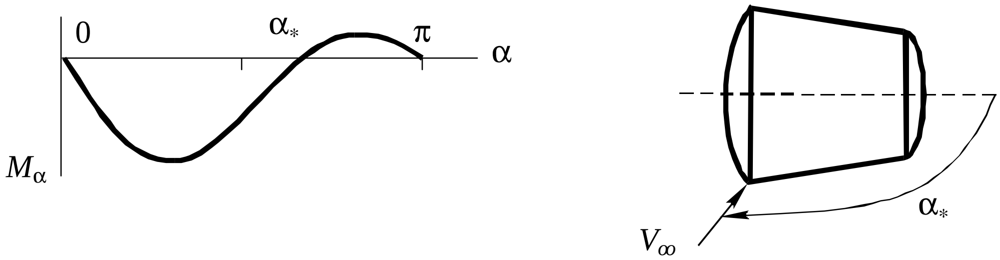

Fig.8. The biharmonic moment characteristic for the capsule having segmentally-conical or blunted conical shape.

Coefficients *a* and *b*, whose variability is associated with the atmospheric density variation during the descent, may be represented in the form [1, 10]

$$a = a_0 z, \quad b = b_0 z, \tag{11}$$

$$a_0 = -m_a S l \rho(t_0) V_0^2 /(2I),$$

$$b_0 = -m_b S \rho(t_0) l V_0^2 /(2I),$$

$$z = \exp(\beta t), \quad \beta = \lambda V_0 \,|\sin\theta_0\,|,$$

where m_a, m_b are constant cœfficients, $\rho(t_0)$ is the atmospheric density in an initial instant, λ is the logarithmic density gradient in height, V_0 is the flight velocity, θ_0 is the trajectory inclination angle.

The energy integral of system (10) for constants a and b has the form of

$$E = \dot{\alpha}^2/2 + W(\alpha) = h, \tag{12}$$

where

$$W(a) = 0.5(R^2 + G^2 - 2RG\cos a)/\sin^2 a -$$
$$-a\cos a - b\cos^2 a$$

- potential energy of a system.

The type of the system's motion is determined by the relation between quantities a, b, R, G, and h. In the planar case of motion ($R = G = 0$), three types phase portraits take place.

(1) $|a| \ge 2|b|$. The phase portrait is analogous to an oscillatory system of the pendulum type and is depicted for $a > 0$ in Fig. 9 (for $a < 0$, the phase picture is shifted by value π along the α-axis).

(2) $b > 0.5|a|,\ b > 0$. Some additional singular points of saddle type appear on the phase portrait. These points correspond to the values of the angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi\ (n = 0, \pm 1, \pm 2, ...)$, and three regions of motion take place: a rotational and two oscillatory ones (Fig. 10).

(3) $|b| > 0.5|a|,\ b < 0$. The phase portrait for the case $a > 0$ is shown in Fig. 11 (for $a < 0$, the phase picture is shifted by value π along the α-axis). Here some singular points of center type correspond to the values of the angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi\ (n = 0, \pm 1, \pm 2, ...)$, and four regions of motion take place, a rotational and three oscillatory ones (Fig. 11).

In the spatial case of motion, the presence of a gyroscopic term in equation (10) stipulates the exclusively oscillatory character of capsule motion. The presence of the second harmonic in the moment characteristics causes a possibility of appearance of some singular point of saddle type on the phase portrait of a system. In this case, there are three oscillation regions (Fig. 12).

In connection with the change of a and b coefficients during the motion, the evolution of phase trajectories takes place. As a result, these trajectories can intersect separatrices and fall into various phase portrait regions, which is followed by qualitative changes in the motion character. Figure 13 shows one of the possible versions of an angle of attack variation in the case of a capsule's spatial motion about the center of mass during descent.

To describe the motion of a system with slowly varying parameters (10), we shall use the integral of action written in the form of

$$I_g = \int_{\alpha_{min}}^{\alpha_{max}} \dot{\alpha}\, d\alpha = \text{const}, \tag{13}$$

where $\alpha_{min}, \alpha_{max}$ are amplitude values of the angle of attack (in a planar rotation, $\alpha_{min} = -\pi,\ \alpha_{max} = \pi$); $\dot{\alpha}$ is determined from the energy integral (12).

For system (10), the equality $I_g = \text{const}$ is valid for the majority of boundary conditions to an accuracy of $O(\varepsilon \ln \varepsilon)$ for times of the order of $1/\varepsilon$ [11], where ε is a small parameter characterizing the rate of the variation of parameter z. An exceptional set of initial conditions, for which this evaluation is invalid, has a measure $O(\varepsilon^n)$, where $n \ge 1$ is any prespecified number. The motion modes corresponding to the given initial conditions are called the modes of capsule hovering in the unstable equilibrium vicinity. These modes were thoroughly investigated in [1] and are not considered here.

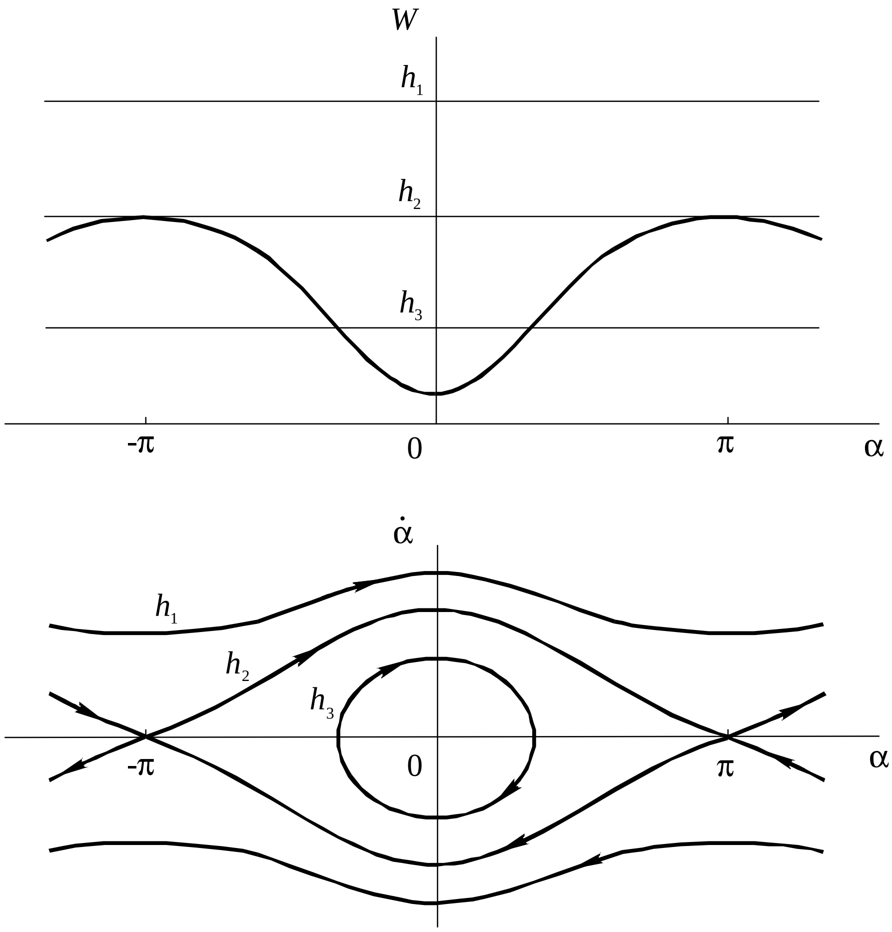

Fig. 9. Potential energy and phase trajectories of the planar motion: ($|a|\geq 2|b|$).

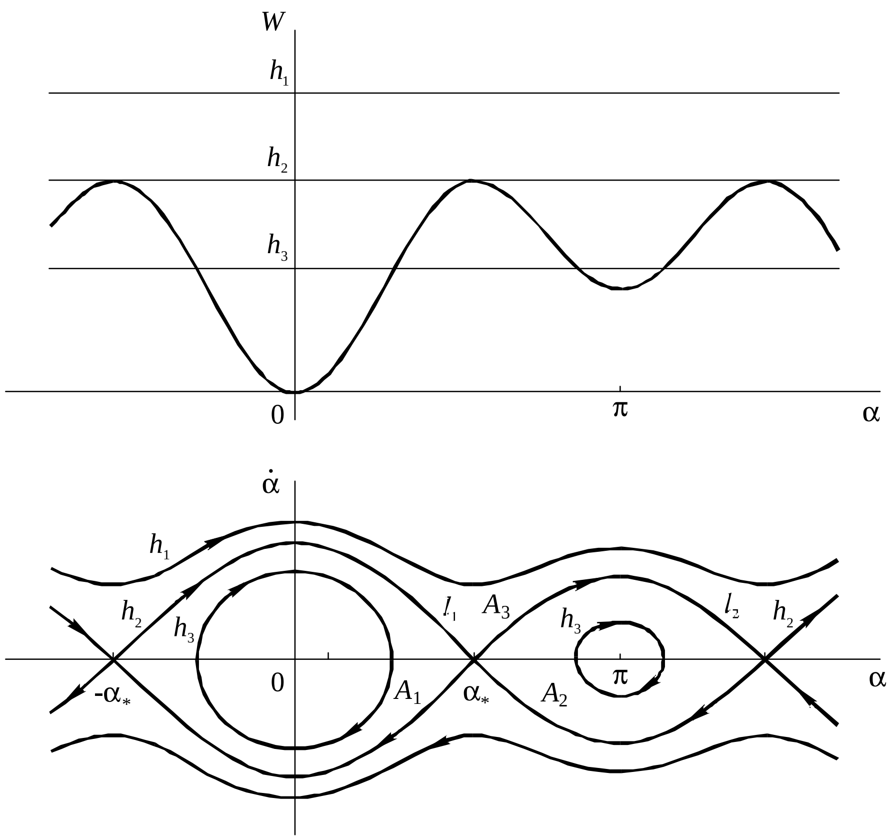

Fig. 10. Potential energy and phase trajectories of the planar motion: ($b>0.5|a|$, $b>0$).

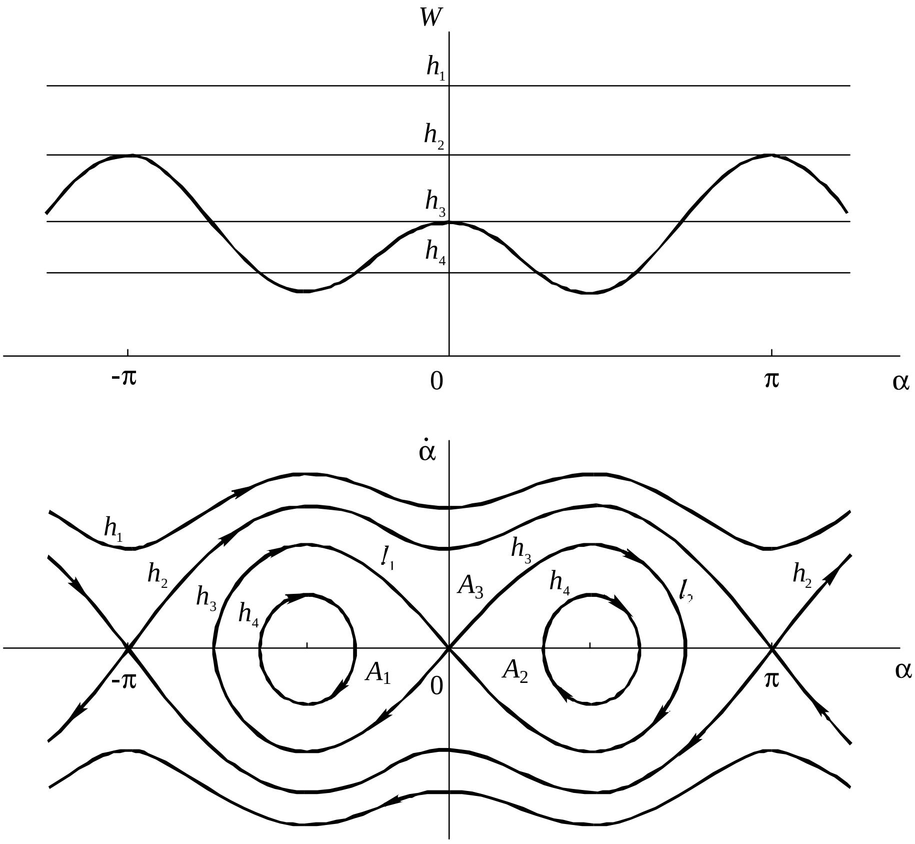

Fig. 11. Potential energy and phase trajectories of the planar motion: ($|b|>0.5|a|$, $b<0$).

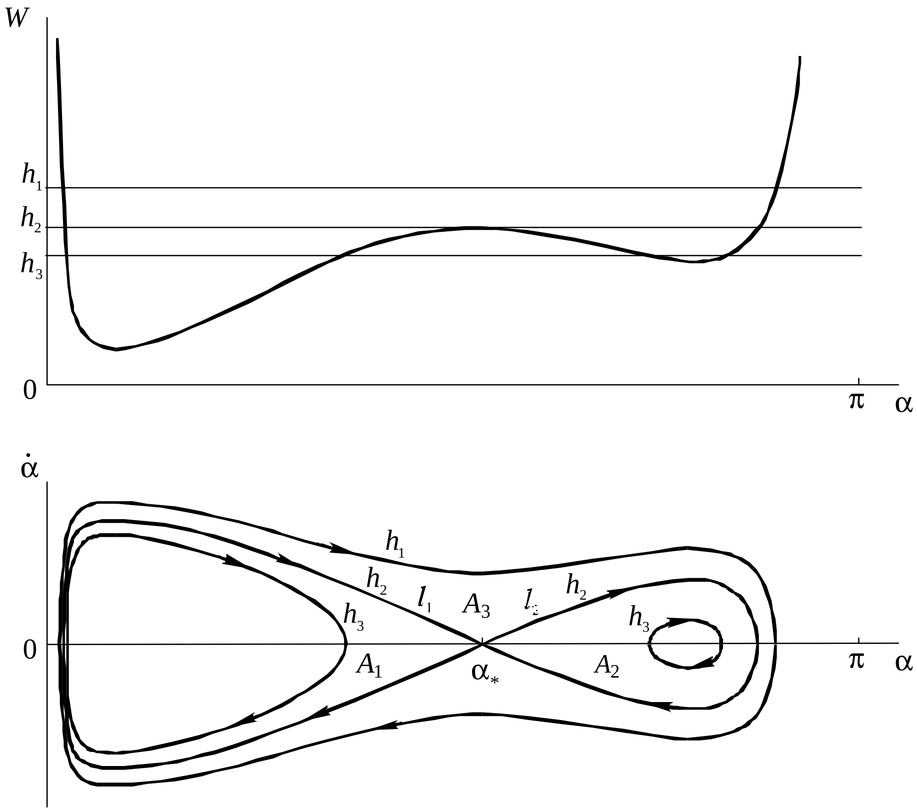

Fig. 12. Potential energy and phase trajectories of the spatial motion.

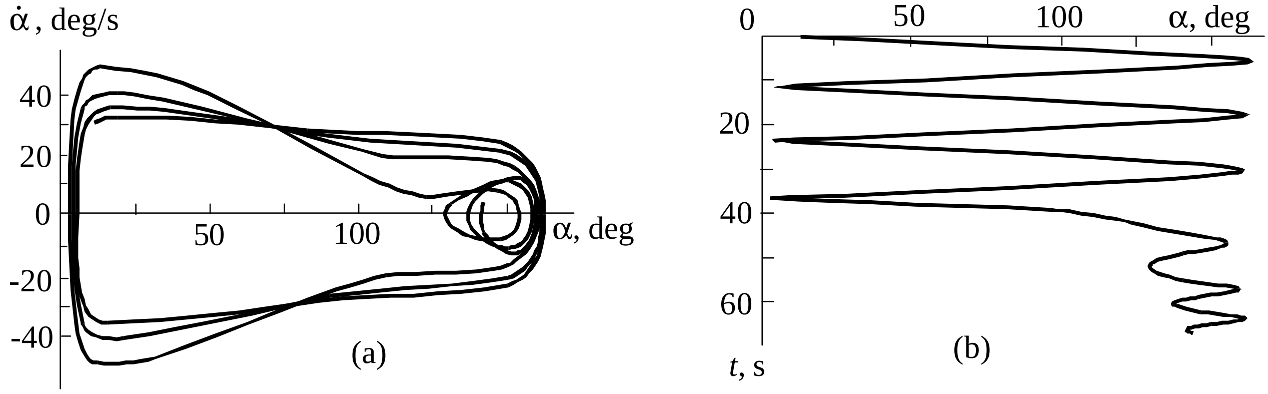

Fig. 13. Character of spatial motion variation during the reentry:
(a) phase trajectories; (b) variation of the angle of attack.

Let us consider a procedure of studying of transient modes of motion of a capsule, when the integration of equations of motion is not carried out, and are used only the analytical formulas for the action integral (13) are obtained in full elliptic integrals or elementary functions [10].

Radiating from persistence of an action integral, the time moments corresponding to the transitions between various phase portrait regions are determined from the equality of the action integral expression calculated along the separatrices to the action integral value calculated from the initial conditions of motion.

$$\mu = \frac{2\dot\alpha}{\lambda V_0 |\sin\theta_0|} >> 1.$$

In cases when, at intersecting separatrices, the phase point may fall into various oscillation regions, the problem of choosing a motion continuation region arises. The point is that phase points which in the initial moment were on distance about ε from each other, can intersect separatrices and fall into various regions, and their further motion will be completely different. As initial data always are known only with some accuracy at determined the approach to a problem loses sense. However it is possible to define and calculate correctly probability of falling in this or that area. Let separatrices l_1 and l_2 separate the inner regions of motion A_1, A_2 from the outer one, A_3 (see Figs. 2, 3, 4). For choosing the motion continuation region A_1, or A_2, the probability P_i, $i = 1, 2$, of capture into each is used. In accordance with [12], this probability is defined as the fraction of a phase volume of a small neighborhood around the initial point of motion, which is "captured" into the region under consideration in the limit when a small parameter $\varepsilon \to 0$ and the dimension of neighborhood $\delta \to 0$, $\varepsilon << \delta$ (the limit is first taken over ε and then, over δ, where) $P_1 + P_2 = 1$. The ratio of probabilities is calculated by formulas

$$\frac{P_1}{P_2} = \frac{\Theta_1}{\Theta_2}, \qquad (14)$$

$$\Theta_i = -\oint_{l_i} \frac{\partial\,[E(\dot\alpha,\alpha,z) - E(0,\alpha_*,z)]}{\partial z} f_z dt \qquad (15)$$

($i = 1, 2$),

where $\dot\alpha = 0$, $\alpha = \alpha_*$ are coordinates of a singular saddle-type point on the phase portrait, $f_z = \dot z = \beta z$. Integrals (15) are calculated along the separatrices l_1 and l_2, parametrized by time t of undisturbed motion along these separatrices.

One should note that, since we consider the upper section of the reentry trajectory for which $f_z > 0$, the quantities Θ_i, will also be positive; therefore, a single passage through the phase point separatrix from the outer region into the inner one takes place [12].

Therefore, with the known initial conditions of motion at the atmosphere boundary, one may trace the phase trajectory evolution, find the times of transition and transition probability into each characteristic region of a phase portrait.

The analysis of transient modes will be carried out for three aforementioned cases of planar motion, the phase portraits of which are shown in Figs. 9-11, and for the case of spatial motion, the phase portrait of which are shown in Fig. 12.

(1) $|a| \ge 2|b|$. The phase portrait is depicted for $a > 0$ in Fig. 9 (for $a < 0$, the phase picture is shifted by value π along the α-axis).

Depending on value of energy h the capsule can make either rotary, or oscillating motion, the regions are divided which one by a separatrix. Let starting conditions correspond to a rotation of a capsule. In process of growth of parameter z the oscillatory region grows, and the capsule making a rotary motion, in certain moment begins to make an oscillating motion.

The time corresponding to the moment of transition of rotation into oscillations is determined from expressions (11):

$$t_* = \ln[b_* / b_0]/\beta \qquad (16)$$

or

$$t_* = \ln[a_* / a_0]/\beta ,$$

where the value of coefficient b or a at time $t = t_*$, corresponding to the moment of transition of rotation into oscillations, is determined on one of the following formulas [10]:

$$b_* = \{I_g^0 /[\sqrt{u_* - 1} + u_* \arctan(\sqrt{1/(u_* - 1)})]\}^2 / 32$$

for $b > 0$,

$$b_* = -\{I_g^0 /[\sqrt{u_* + 1} + u_* \ln((1 + \sqrt{u_* + 1})/\sqrt{u_*})]\}^2 / 32$$

for $b < 0$, (17)

$$a_* = \mathrm{sign}(a_0)(I_g^0)^2 / 64 \quad \text{for } b = 0,$$

where $u_* = | 0.5a_* / b_* | = | 0.5a_0 / b_0 |$.

Here the action integral I_g^0 is calculated from the initial conditions of motion pursuant to (13) or under the analytical formulas derived in [10]. In the case when, at the atmosphere boundary, a_0 and b_0 coefficients are essentially small as compared to the angular velocity $\dot{\alpha}_0$, the action integral can be determined by formula

$$I_g^0 = 2\pi\dot{\alpha}_0 .$$

(2) $b > 0.5|a|,\ b > 0$. The phase portrait is shown in Fig. 10. The picture of phase curves is periodic on α with phase 2π.

The values of an angle of attack $\theta = n\pi$ $(n = 0, \pm 1, \pm 2, ...)$, corresponding to singular points of a type center, are stable equilibrium positions. The values of an angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi$ $(n = 0, \pm 1, \pm 2, ...)$, corresponding to singular points of a type saddle, are unstable equilibrium positions. Three regions of motion on the phase portrait take place: a rotational A_3 and two oscillatory A_1 and A_2.

Depending on value of energy h the capsule can make either rotary, or oscillating motion. In process of growth of parameter z the oscillatory regions A_1 and A_2 grows, and the capsule making a rotary motion, in certain moment begins to make an oscillating motion in one of them.

The time corresponding to the transition of rotation into oscillations is calculated by formula (16), and in this case, coefficient b_* is determined by formula

$$b_* = [I_g^0 /(\sin\alpha_* + (0.5\pi - \alpha_*)\cos\alpha_*)]^2 / 32, \qquad (18)$$

where $\alpha_* = \arccos(-0.5a_0 / b_0)$.

Since at $t > t_*$ the capsule can oscillate with respect to one of two stable positions of equilibrium in the angle of attack $\alpha = 0$ or $\alpha = \pi$, let us determine the probability of falling into these oscillation regions. We denote by P_1 the probability of falling into the vicinity of the angle of attack value $\alpha = 0$; by P_2, the probability of falling into the vicinity of $\alpha = \pi$, where $P_1 + P_2 = 1$. After appropriate calculations by formulas (14) and (15), we have [10]

$$\frac{P_1}{P_2} = \frac{1 - \alpha_* \cot\alpha_*}{1 + (\pi - \alpha_*)\cot\alpha_*}. \qquad (19)$$

As seen, the value of the probability of the capsule falling into any oscillation region is determined only by the value of an unstable position of equilibrium in the angle of attack $\alpha = \alpha_*$.

(3) $|b| > 0.5|a|,\ b < 0$. The phase portrait for the case $a > 0$ is shown in Fig. 11 (for $a < 0$, the phase picture is shifted by value π along the α-axis). The picture of phase curves is periodic on α with phase 2π. The values of an angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi$ $(n = 0, \pm 1, \pm 2, ...)$, corresponding to singular points of a type center, are stable equilibrium positions. The values of an angle of attack $\theta = n\pi$ $(n = 0, \pm 1, \pm 2, ...)$, corresponding to singular points of a type saddle, are unstable equilibrium positions. Four regions of motion on the phase portrait take place: a rotational and three oscillatory.

Depending on value of energy h the capsule can make either rotary, or oscillating motion. The capsule can oscillate or with respect to unstable equilibrium position $\alpha = n\pi$ $(n = 0, \pm 1, \pm 2, ...)$ in region A_3 or with respect to one of two stable positions of equilibrium $\alpha = \pm\arccos(-0.5a/b) + 2n\pi$ $(n = 0, \pm 1, \pm 2, ...)$ in region A_1 or A_2.

In process of growth of parameter z the oscillatory regions grows, and the capsule making a rotary motion, in certain moment begins to make an oscillating motion with respect to unstable equilibrium position, and then in a following moment begins to make an oscillating motion with respect to one of two stable positions of equilibrium.

The time corresponding to the transition of rotation into oscillations is calculated by formula (16), and in this case, coefficient b_* is determined by formula (17).

The time corresponding to the transition from the region of oscillations with respect to an unstable equilibrium position $\alpha = 0$ (for $a > 0$) or $\alpha = \pi$ (for $a < 0$) into one of two regions of oscillations with respect to stable equilibrium positions

$\alpha = \pm\arccos(-0.5a/b)$ (in region A_2 or A_1) is calculated by formula (16). In this case, coefficient b_* is determined by formula

$$b_* = -\{I_g^0/[\sqrt{1-u_*} + u_*\ln((1+\sqrt{1-u_*})/\sqrt{u_*})]\}^2/32 \quad (20)$$

where $u_* = |0.5a_*/b_*| = |0.5a_0/b_0|$.

The falling into the regions of oscillations A_1 or A_2 is equally probable, since the oscillation regions are equal and symmetrical with respect to a singular saddle point $\alpha = 0$ (for $a > 0$) or $\alpha = \pi$ (for $a < 0$).

The phase portrait for equation (10) in the case of a spatial motion is determined by the relation between quantities h, a, b, R and G. The qualitative analysis of equation (10) shows that, if, inside the interval for the angle of attack $(0, \pi)$, the saddle point is absent in the planar case of $R = G = 0$, then it is also absent in the case of spatial motion irrespective of quantities R and G. On the other hand, if, for $R = G = 0$, the saddle point actually takes place (the case of $b > 0.5|a|,\ b > 0$), then its absence can be provided only by choosing sufficiently high (in magnitude) and finite R and G values. We analyze the case when the saddle point exists; then the phase portrait of equation (10) has the form shown in Fig. 12.

Depending on value of energy h the capsule can make either oscillating motion in outer region A_3, or oscillating motion in one of two inner regions A_1 or A_2 (Fig. 12).

In process of growth of parameter z the oscillatory regions grows, and the capsule making a oscillating motion in outer region A_3, in certain moment begins to make an oscillating motion in one of two inner regions A_1 or A_2.

For an illustration of the data of transient modes of motion on Fig. 13 the character of variation of spatial motion of a capsule during the reentry in a case is exhibited, when the motion is terminated in region A_2 (initial data: $a_0 = -0.02\ \mathrm{s}^{-2}$, $b_0 = -0.02\ \mathrm{s}^{-2}$, $R = 0.05$?g $\cdot\ \mathrm{m}^2/\mathrm{s}$, $G = 0.1$?g $\cdot\ \mathrm{m}^2/\mathrm{s}$, $\alpha_0 = 10\ \mathrm{deg}$, $\dot{\alpha} = 27\ \mathrm{deg/s}$, $\beta = 0.05\ \mathrm{s}^{-1}$).

The time corresponding to the transition from an outer oscillation region A_3 into one of inner oscillation regions is calculated by formula (16). In this case, coefficient b_* is determined by formula [10]

$$b_* = \left\{ I_g^0 \Big/ \left[2\sqrt{(u_1-u_*)(u_*-u_2)} - \sum_{i=1}^{3} c_i \arcsin\delta_i \right] \right\} \quad (21)$$

where $c_1 = u_1 + u_2 + 2u_*$,

$c_{2,3} = (1 \mp u_*)\sqrt{(u_1 \mp 1)(u_2 \mp 1)}$,

$\delta_1 = (u_1 + u_2 - 2u_*)/|u_2 - u_1|$,

$d_{2,3} = [(u_2 \mp 1)(u_* - u_1) + (u_1 \mp 1)(u_* - u_2)] / / |(u_2 - u_1)(u_* \mp 1)|$;

$u_1 = \cos\alpha_{\min}$, $u_2 = \cos\alpha_{\max}$, $u_3 = u_4 = u_*$ are the roots of polynomial:

$$f(u) = -2bu^4 - 2au^3 + 2(b-h)u^2 + 2(a+GR)u + +(2h - G^2 - R^2)$$

in this case, quantity h is determined from the condition of transition of complex-conjugated roots $u_{3,4} = u_{34} \pm iw$ into real ones $u_{3,4} = u_{34} = u_* = \cos\alpha_*$, $w = 0$.

After intersecting separatrices the capsule at $t > t_*$ may continue its motion in one of two inner oscillation regions A_1 or A_2 (Fig. 12). The probability of capturing into these oscillation regions is determined by formula [10]

$$\frac{P_1}{P_2} = \frac{\sqrt{(u_1-u_*)(u_*-u_2)} + (0.5c_1 + a_0/b_0)(0.5\pi + \arcsin\delta_1)}{\sqrt{(u_1-u_*)(u_*-u_2)} - (0.5c_1 + a_0/b_0)(0.5\pi - \arcsin\delta_1)}$$

Thus, we have investigated the transient modes of motion of a spacecraft with a biharmonic moment characteristic at the upper section of a trajectory both in a planar and in a spatial case of motion. The formulas for determining the moments of transition between various phase plane regions are presented. For the cases of motion, when intersecting a separatrix, the phase point may fall into various oscillation regions; the formulas are presented for determining the probability of capture into any region.

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